Stability of stochastic systems driven by Lévy processes

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Outline

Background of Lévy process

- The basic theory was established in the 1930s by Paul Lévy, but a great deal of new theoretical development as well as novel applications have appeared in recent years.
- They form special subclasses of both semimartingales and Markov processes.
- They are the simplest examples of random motions whose sample paths are right-continuous and have a number (at most countable) of random jump discontinuities occurring at random times on each finite time interval.

Background of Lévy process

• Mathematical finance, the fluctuations in the financial markests, quantum field theory, filtering, economics, control, physics, mechanics, engineering;

• Earthquakes, hurricanes, epidemics;

• Important examples: Brownian motion and Gaussian processes, Poisson processes, Compound Poisson processes, Interlacing processes, α-stable Lévy process, Subordinators.

The definition of Lévy process

Definition 1(Lévy process)
Let $L(t)$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, then $L(t)$ is a Lévy process if the following conditions hold:
(a) $L(0) = 0 \quad (a.s.)$;
(b) $L(t)$ has independent and stationary increments;
(c) $L(t)$ is stochastically continuous,
i.e. for all $k > 0$ and $t, s \geq 0$,

$$\lim_{t \to s} P(|L(t) - L(s)| > k) = 0;$$

Remark

Remark 1 In the presence of (a) and (b), (c) is equivalent to the following condition: for all $k > 0$,

$$\lim_{t \downarrow 0} P(|L(t)| > k) = 0.$$

Remark 2 $L(t)$ has c*à*dl*à*g paths (i.e., the paths are right continuous with left limits.).

Some examples of Lévy processes

1. **Brownian motion:** A standard Brownian motion $B = (B(t), t \geq 0)$ is a Lévy process on $\mathbb{R}^d$ such that

• $B(t) \sim N(0, tI)$ for each $t \geq 0$,

• B has continuous sample paths.

The characteristic function of a standard Brownian motion is given by

$$\phi_{B(t)}(u) = exp\left\{-\frac{1}{2}t|u|^2\right\},$$

for each $u \in \mathbb{R}^d, t \geq 0$.

Some examples of Lévy processes

2. **Poisson process:** Let $N = (N(t), t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. It is a Lévy process with non-negative integer values. For each $t > 0, N(t)$ follows a Poisson distribution with parameter λt so that

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

for each $n = 0, 1, 2, ...$

Remark

A Poisson process has piecewise constant paths on each finite time interval and at the random times

$$\tau_n = \inf\{t \geq 0 : N(t) = n\},$$

and it has jumps of size 1.
Its characteristic function is given by

$$\phi_{N(t)}(u) = exp\left[\lambda t(e^{iu} - 1)\right],$$

for each $u \in \mathbb{R}^d, t \geq 0$.

Some examples of Lévy processes

3. **Compound Poisson process:**

Let $N = (N(t), t \geq 0)$ be a Poisson process with intensity $\lambda > 0$ and let $(U(m), m \in N)$ be a sequence of independent, identically distributed random variables on $\mathbb{R}^d$, independent of N and defined on the probability space $(\Omega, \mathcal{F}, P)$ with common law μ_U.

The compound Poisson process $Z = (Z(t), t \geq 0)$ is a Lévy process, where for each $t \geq 0$,

$$Z(t) = \sum_{k=1}^{N(t)} U(k).$$

Some examples of Lévy processes

The characteristic function of a compound Poisson process is given by

$$\phi_{Z(t)}(u) = exp\left\{t\left[\int_{\mathbb{R}^d}(e^{<iu,y>} - 1)\lambda_{\mu U}(y)\right]\right\},$$

for each $u \in \mathbb{R}^d, t \geq 0$ and we can deduce that the Lévy measure for Z is $\lambda_{\mu U}$.

Lévy-Khintchine formula

If $L(t)$ is a Lévy process, then for all $t \geq 0$ and $u \in \mathbb{R}^d$,

$$\mathbf{E}(e^{i(u,L(t))}) = e^{t\eta(u)},$$

where

$$\eta(u) = i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - \{0\}} [e^{i(u,y)} - 1 - i(u,y)I_{(0<|y|<1)}(y)]\nu(dy),$$

and A is a positive definite symmetric $d \times d$ matrix.

The law of $L(t)$ is uniquely determined by its characteristics (b, A, ν).

Lévy Itó decomposition Theorem

If $L(t)$ is a Lévy process, then there exist a constant $b \in \mathbb{R}^d$, a Brownian motion B_A with covariance matrix A and independent Poisson random measure N defined on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ with compensator $\tilde{N}$, such that for every $t \geq 0$,

$$L(t) = bt + B_A(t) + \int_{|y|<c} y\widetilde{N}(t, dy) + \int_{|y|\geq c} yN(t, dy),$$

where

- $\widetilde{N}(t, dy) = N(t, dy) - t\nu(dy)$,

• N is independent of B and is a Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ with compensator $\tilde{N}$ and intensity measure ν.

• ν is a Lévy measure defined on $\mathbb{R}^d - \{0\}$, which satisfies

$$\int_{\mathbb{R}^d - \{0\}} (y^2 \wedge 1)\nu(dy) < \infty,$$

• $b = E(L(1) - \int_{|y| \geq c} yN(1, dy))$,

• the parameter $c \in [0, \infty)$ is a constant.

•Usually, the pair (B, N) is called a *Lévy noise*.

Lévy martingale

• The process $L(t)$ is a martingale, and will be called a Lévy martingale, if and only if

$$\int_{(|x|>1)} |x|\nu(dx) < \infty \quad \text{and} \quad b = -\int_{(|x|>1)} x\nu(dx).$$

In this case, the process $L(t)$ can be written as follows:

$$L(t) = B_A(t) + \int_{\mathbb{R}^d - \{0\}} y\widetilde{N}(t, dy).$$

• If $L(t) = L_M(t) + bt$, where L_M is a Lévy martingale, we call $L(t)$ a Lévy martingale with drift.

Stochastic system of Lévy processes:

Consider the following stochastic differential equation driven by Lévy processes:

$$dx(t) = f(x(t))dt + g(x(t))dL(t), \; t \geq t_0 \geq 0, \tag{1}$$

with the initial value $x(t_0) = x_0 \in \mathbb{R}^d$, where f and g are two functions, and $L(t)$ is a Lévy process.

Stochastic system of Lévy processes:

By using the Lévy Itó decomposition Theorem, we establish the following model:

$$dx(t) = f(x(t))dt + g(x(t))dB(t) + \int_{|y|<c} h_1(x(t-), y)\tilde{N}(dt, dy) + \int_{|y|\geq c} h_2(x(t-), y)N(dt, dy), \ t \geq t_0 \geq 0, \tag{2}$$

with the initial value $x(t_0) = x_0 \in \mathbb{R}^d$, where the mappings $f : \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R}), h_i(i = 1, 2) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, and the constant $c \in (0, \infty]$ is the maximum allowable jump size.

Here, $\mathcal{M}_{d,m}$ denotes the space of all real-valued $d \times m$ matrices with the norm $\|A\| := (\sum_{i=1}^{d} \sum_{j=1}^{m} |a_{ij}a_{ji}|)^{\frac{1}{2}}$, $A = (a_{ij})_{d\times m}$.

Conditions:

Assumption 1 f, g, h_1, h_2 satisfy the global Lipschitz condition, i.e., there exist four positive constants $K_i (i = 1, 2, 3, 4)$ such that

$$|f(x_1) - f(x_2)| \leq K_1|x_1 - x_2|, \quad |g(x_1) - g(x_2)| \leq K_2|x_1 - x_2|,$$

$$\int_{|y|<c} |h_1(x_1, y) - h_1(x_2, y)|\nu(dy) \leq K_3|x_1 - x_2|,$$

$$\int_{|y|\geq c} |h_2(x_1, y) - h_2(x_2, y)|\nu(dy) \leq K_4|x_1 - x_2|,$$

for all $x_1, x_2 \in \mathbb{R}^d$.

Assumption 2 $f(0) = 0, g(0) = 0$, $h_1(0, y) = 0$ for all $|y| < c$, and $h_2(0, y) = 0$ for all $|y| \geq c$.

Existence and Uniqueness Theorem (I):

Remark 2 Under Assumptions 1 and 2, Theorem 1 below ensures that (2) has a unique solution $x(t)$. In particular, $x(t) \equiv 0$ for all $t \geq t_0$ corresponding to the initial data $x(t_0) = 0$, which is often called the *trivial solution*.

Theorem 1 Under Assumptions 1 and 2, there exists a unique solution $x(t)$ to Eq. (2) with the standard initial condition. Moreover, the solution $x(t)$ is equals to zero for all $t \geq t_0$ corresponding to the initial data $x(t_0) = 0$, which is often called the *trivial solution*.

Proof. By using the well-known Picard iteration, together with Doob's martingale inequality, Cauchy-Schwarz inequality, Gronwall's inequality, Itô's isometry and stochastic analysis theory.

Conditions:

Assumption 3 f, g, h_1, h_2 satisfy the local Lipschitz condition, i.e., for each $m = 1, 2, ...,$, there exist four positive constants $K_{im}, i = 1, 2, 3, 4$ such that

$$|f(x_1) - f(x_2)| \leq K_{1m}|x_1 - x_2|, \quad |g(x_1) - g(x_2)| \leq K_{2m}|x_1 - x_2|,$$

$$\int_{|y|<c} |h_1(x_1, y) - h_1(x_2, y)|\nu(dy) \leq K_{3m}|x_1 - x_2|,$$

$$\int_{|y|\geq c} |h_2(x_1, y) - h_1(x_2, y)|\nu(dy) \leq K_{4m}|x_1 - x_2|,$$

for all $x_1, x_2 \in \mathbb{R}^d$ with $|x_1| \vee |x_2| \leq m$, where $a \vee b$ represents $\max\{a, b\}$.

Remark:

Remark 3 Under Assumptions 2 and 3, Eq. (2) has a unique local solution $x(t)$. To ensure a unique global solution $x(t)$ to Eq. (2), we need an additional condition. To present the condition, we need to the following infinitesimal generator A of $x(t)$ defined by

$$Af(x) := \lim_{t \downarrow 0} \frac{E^x[f(x(t))] - f(x)}{t}, \quad x \in \mathbb{R}^d, \tag{3}$$

where the set of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_A(x)$, while $\mathcal{D}_A$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^d$.

Notations:

• $C_1^2(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ which are continuously twice differentiable in x and differentiable in t.

$$V_t(t,x(t)) = \frac{\partial V(t,x(t))}{\partial t}, \quad V_{xx}(t,x(t)) = \left(\frac{\partial^2 V(t,x(t))}{\partial x_i \partial x_j}\right)_{n\times n},$$
$$V_x(t,x(t)) = \left(\frac{\partial V(t,x(t))}{\partial x_1}, \ldots, \frac{\partial V(t,x(t))}{\partial x_n}\right).$$

• $\mathcal{B}_h := \{x \in \mathbb{R}^d : |x| < h\}$
• $C_1^2(\mathbb{R}_+ \times \mathcal{B}_h; \mathbb{R}_+)$ can be defined similarly.

Define an operator $\mathcal{G}V$:

If $V \in C_1^2(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+)$, then according to [D. Applebaum. (2009)], we can define an operator $\mathcal{G}V$ from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}$ by

$$\begin{aligned}
\mathcal{G}V(t,x(t)) = {} & V_t(t,x(t)) + V_x(t,x(t))f(x(t)) \\
& + \frac{1}{2} trace[g^T(x(t))V_{xx}(t,x(t))g(x(t))] \\
& + \int_{|y|<c} [V(t,x(t)+h_1(x(t),y)) - V(t,x(t)) \\
& - h_1(x(t),y)V_x(t,x(t))]\nu(dy) \\
& + \int_{|y|\geq c} [V(t,x(t)+h_2(x(t),y)) - V(t,x(t))]\nu(dy).
\end{aligned} \tag{4}$$

Remark 4 Note that $\mathcal{G}$ is the infinitesimal generator of the Feller semigroup when $\{x(t), t \geq t_0\}$ is a Feller process.

Conditions:

Define $U_R^c := \{x \in \mathbb{R}^d : |x| > R\}$

Assumption 4 There exists a nonnegative function $V(t,x)$ on $\mathbb{R}_+ \times U_R^c$ that is twice continuously differentiable in $x \in U_R^c$ for some $R > 0$ sufficiently large and differentiable in t, such that there exists a constant $\alpha > 0$ satisfying

$$\mathcal{G}V(t,x(t)) \leq \alpha V(t,x(t)),$$
$$\inf_{x(t)>R} V(t,x(t)) \to \infty \quad \text{as} \quad R \to \infty.$$

Existence and Uniqueness Theorem (II):

Theorem 2 Under Assumptions 2,3 and 4, there exists a unique global solution $x(t)$ to Eq. (2) with the standard initial condition.

Proof. Under Assumptions 2 and 3, there exists a unique maximal solution $x = \{x(t), 0 \leq t < \sigma\}$, where σ is the explosion time. If we can prove $\sigma = \infty$, we know that $x(t)$ is global. To this end, we define the stopping time

$$\sigma_k = \inf\{0 \leq t < \sigma : V(t, x(t)) \geq k\}.$$

Obviously, σ_k is increasing on k, and so we have $\sigma_k \to \sigma_\infty$ as $k \to \infty$. Furthermore, we have $\sigma_\infty \leq \sigma$. So we only need to prove $\sigma_\infty = \infty$.

Stochastic stability:

Definition 1 The trivial solution of (2) is said to stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $\rho > 0$, there exists a $\delta = \delta(\varepsilon, \rho, t_0)$ such that

$$P\{|x(t)| < \rho \quad \text{for all } t \geq t_0\} \geq 1 - \varepsilon$$

whenever $|x_0| < \delta$. Otherwise it is said to be stochastically unstable.

Definition 2 The trivial solution of (2) is said to be almost surely stable if

$$\lim_{t \to \infty} x(t) = 0 \quad a.s.$$

for all $x_0 \in \mathbb{R}^d$.

Stochastic stability:

Definition 3 The trivial solution of (2) is said to be asymptotically stable in the *p*th moment if

$$\lim_{t\to\infty} \mathbf{E}|x(t)|^p = 0$$

for all $x_0 \in \mathbb{R}^d$.

Definition 4 The trivial solution of (2) is said to be *p*th moment exponentially stable if there is a pair of constants $\alpha > 0$ and $\beta > 0$ such that

$$\mathbf{E}|x(t)|^p \leq \alpha |x_0|^p exp\{-\beta(t - t_0)\}$$

for all $t \geq t_0$ and $x_0 \in \mathbb{R}^d$.

Stochastic stability:

Remark 5 We remark that asymptotically stable in the pth moment is often called asymptotically stable in the mean square when $p = 2$.
Similarly, exponentially stable in the pth moment is often called exponentially stable in the mean square when $p = 2$.

Definition 5 The trivial solution of (2) is said to be is said to be almost surely pth moment exponentially exponentially stable if

$$\limsup_{t \to \infty} \frac{1}{t} log|x(t)|^p = 0 \quad a.s.$$

for all $x_0 \in \mathbb{R}^d$.

Stochastic stability:

Definition 6 The trivial solution of (2) is said to be continuous in probability if for any $\varepsilon > 0$ and $x_0 \in \mathbb{R}^d$,

$$\lim_{\delta \to 0} P\{|x(t+\delta) - x(t)| \geq \varepsilon) = 0.$$

Definition 7 The trivial solution of (2) is said to be continuous in the *p*th moment if

$$\lim_{\delta \to 0} \mathbf{E}|x(t+\delta) - x(t)|^p = 0$$

for all $x_0 \in \mathbb{R}^d$. In particular, continuous in the 2th moment is usually called continuous in the mean square when $p = 2$.

Main results

Theorem 3 Under Assumptions 1 and 2, the trivial solution of (2) is stable in probability if there exists a positive definite function $V \in C_1^2(\mathbb{R}_+ \times \mathcal{B}_h; \mathbb{R}_+)$ such that

$$\mathcal{G}V(t, x(t)) \leq 0 \tag{5}$$

for all $x(t) \in \mathcal{B}_h$.
Proof. Take $0 < c < \frac{h}{2}$ and define the stopping time

$$\tau = \inf\{t \geq t_0 : |x(t)| \geq c\}.$$

The main tools of the proof are Itô's formula, stochastic analysis, probability inequalities techniques.

Assumption 5

One of the following two conditions holds:
(i) If $0 < p < 2$, then for any $0 < q \leq p$, there exist positive constants $\tilde{K}_i (i = 1, 2)$ such that

$$\int_{|y|<c} |h_1(x,y)|^q \nu(dy) \leq \tilde{K}_1 |x|^q, \quad (6)$$

$$\int_{|y|\geq c} |h_2(x,y)|^q \nu(dy) \leq \tilde{K}_2 |x|^q, \quad (7)$$

for all $x \in \mathbb{R}^d$;
(ii) If $p \geq 2$, then for any $2 \leq q \leq p$, there exist positive constants $\tilde{K}_i (i = 1, 2)$ such that (6) and (7) are satisfied for all $x \in \mathbb{R}^d$.

Remark:

Remark 6 Assumption 5 is slightly weaker than the following condition:

(H) For any $0 < q \leq p$, there exist positive constants $\tilde{K}_i (i = 1, 2)$ such that (6) and (7) are true for all $x \in \mathbb{R}^d$.

Actually, when $q = 0.5, p = 3$, (6) and (7) are required to hold in **(H)**, but (6) and (7) are allowed to fail in Assumption 5.

Notations

- $\mathcal{K}_\triangle$ denotes the family of all continuous increasing convex functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$ while $\kappa(u) > 0$ for $u > 0$;
- $L^1(\mathbb{R}_+; \mathbb{R}_+)$ is the family of functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \gamma(t)dt < \infty$;
- $\Psi(\mathbb{R}_+; \mathbb{R}_+)$ is the family of continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\delta > 0$ and any increasing sequence $\{t_k\}_{k\geq 1}, \sum_{k=1}^\infty \int_{t_k}^{t_k+\delta} \psi(t)dt = \infty$.

Main results

Theorem 4 Under Assumptions 1, 2 and 5, the trivial solution of (2) is asymptotically stable in the pth moment if there exist functions $V \in C_1^2(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+), \gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+), \kappa_1, \kappa_2 \in \mathcal{K}_\triangle, \psi \in \Psi(\mathbb{R}_+; \mathbb{R}_+)$ and a positive constant $p > 0$ such that

$$V(t, x(t)) \geq \kappa_1(|x(t)|^p), \tag{8}$$

$$\mathcal{G}V(t, x(t)) \leq \gamma(t) - \psi(t)\kappa_2(|x(t)|^p) \tag{9}$$

for all $x(t) \in \mathbb{R}^d$.

Main results

Theorem 5 Under Assumptions 1, 2 and 5, the trivial solution of (2) is asymptotically stable in the pth moment if there exist two functions $V \in C_1^2(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+), \gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$ and two positive constants $p > 0, \alpha > 0$ such that

$$\mathcal{G}V(t, x(t)) \leq \gamma(t) - \alpha |x(t)|^p \tag{10}$$

for all $x(t) \in \mathbb{R}^d$.

Proof of Theorem 5

Proof. By Itô's formula, we have that for any $t > s$,

$$
\begin{aligned}
|x(t)|^p = {}& |x(s)|^p + \int_s^t p|x(u-)|^{p-2}x^T(u-)g(x(u-))dB(u) \\
&+ \int_s^t p|x(u-)|^{p-2}x^T(u-)f(x(u-))du \\
&+ \frac{1}{2}p(p-2)\int_s^t |x(u-)|^{p-4}|x^T(u-)g(x(u-))|^2du \\
&+ \frac{p}{2}\int_s^t |x(u-)|^{p-2}|g(x(u-))|^2du
\end{aligned}
$$

Proof of Theorem 5

$$
\begin{aligned}
&+\int_s^t \int_{|y|<c} [|x(u-)+H(x(u-),y)|^p - |x(u-)|^p]\tilde{N}(du,dy) \\
&+\int_s^t \int_{|y|\geq c} [|x(u-)+I(x(u-),y)|^p - |x(u-)|^p]N(du,dy) \\
&+\int_s^t \int_{|y|<c} [|x(u-)+H(x(u-),y)|^p - |x(u-)|^p \\
&-p|x(u-)|^{p-2}x^T(u-)H(x(u-),y)]\nu(dy)du. \qquad (11)
\end{aligned}
$$

Proof of Theorem 5

$$\leq |x(s)|^p + [K_1 p + \frac{1}{2}p(p-2)K_1 + \frac{p}{2}K_1^2] \int_s^t |x(u-)|^p du$$

$$+ \int_s^t p|x(u-)|^{p-2} x^T(u-) g(x(u-)) dB(u)$$

$$+ \int_s^t \int_{|y|<c} [|x(u-) + H(x(u-),y)|^p - |x(u-)|^p] \tilde{N}(du,dy)$$

$$+ \int_s^t \int_{|y|<c} [|x(u-) + H(x(u-),y)|^p - |x(u-)|^p$$

$$-p|x(u-)|^{p-2} x^T(u-) H(x(u-),y)] \nu(dy) du +$$

$$\int_s^t \int_{|y|\geq c} [|x(u-) + I(x(u-),y)|^p - |x(u-)|^p] N(du,dy). \quad (12)$$

Proof of Theorem 5

Thus, for any $\delta \geq 0$, we get

$$\mathbf{E}\left(\sup_{s\leq t\leq s+\delta}|x(v)|^p\right)$$

$$\leq \mathbf{E}|x(s)|^p + [K_1 p + \frac{1}{2}p(p-2)K_1 + \frac{p}{2}K_1^2]\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du$$

$$+\mathbf{E}\left[\sup_{s\leq t\leq s+\delta}\int_s^t p|x(u-)|^{p-2}x^T(u-)g(x(u-))dB(u)\right]$$

Proof of Theorem 5

$$
\begin{aligned}
&+\mathbf{E}\{\sup_{s\le t\le s+\delta}[\int_s^t\int_{|y|<c}(|x(u-)+H(x(u-),y)|^p\\
&-|x(u-)|^p)\tilde{N}(du,dy)+\int_s^t\int_{|y|<c}(|x(u-)+H(x(u-),y)|^p\\
&-|x(u-)|^p-p|x(u-)|^{p-2}x^T(u-)H(x(u-),y))\nu(dy)du]\}\\
&+\mathbf{E}\{\sup_{s\le t\le s+\delta}[\int_s^t\int_{|y|\ge c}(|x(u-)+I(x(u-),y)|^p\\
&-|x(u-)|^p)\tilde{N}(du,dy)+\int_s^t\int_{|y|\ge c}(|x(u-)+I(x(u-),y)|^p\\
&-|x(u-)|^p)\nu(dy)du]\}. \qquad (13)
\end{aligned}
$$

Proof of Theorem 5

By the Burkholder-Davis-Gundy inequality (for instance, see Mao (1997,2008) pp. 129), we have

$$\mathbf{E}\left[\sup_{s\le t\le s+\delta}\int_s^t p|x(u-)|^{p-2}x^T(u-)g(x(u-))dB(u)\right]$$
$$\le \frac{1}{2}\mathbf{E}\left[\sup_{s\le t\le s+\delta}|x(v-)|^p\right]+16p^2K_1\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du.$$

Proof of Theorem 5

On the other hand, from the Kunita's estimate (see Applebaum (2004, 2009), Kunita(1984)) and Hölder inequality we see that there exist two positive constants $c_1(p)$ and $c_2(p)$ such that

$$\begin{aligned}
&\mathbf{E}\{\sup_{s\le t\le s+\delta}[\int_s^t\int_{|y|<c}(|x(u-)+H(x(u-),y)|^p-|x(u-)|^p)\tilde{N}(du,dy)\\
&\quad+\int_t^v\int_{|y|<c}(|x(u-)+H(x(u-),y)|^p-|x(u-)|^p\\
&\quad-p|x(u-)|^{p-2}x^T(u-)H(x(u-),y))\nu(dy)du]\}\\
&\le c_1(p)\mathbf{E}\left[\left(\int_s^{s+\delta}\int_{|y|<c}|H(x(u-),y)|^2\nu(dy)du\right)^{\frac{p}{2}}\right]\\
&\quad+c_2(p)\mathbf{E}\left[\left(\int_s^{s+\delta}\int_{|y|<c}|H(x(u-),y)|^p\nu(dy)du\right)\right]
\end{aligned}$$

Proof of Theorem 5

$$\leq c_1(p)\delta^{\frac{p}{2}-1}\mathbf{E}\left[\int_s^{s+\delta}\left(\int_{|y|<c}|H(x(u-),y)|^2\nu(dy)\right)^{\frac{p}{2}}du\right]$$
$$+c_2(p)\mathbf{E}\left[\left(\int_s^{s+\delta}\int_{|y|<c}|H(x(u-),y)|^p\nu(dy)du\right)\right]$$
$$\leq\left[c_1(p)\delta^{\frac{p}{2}-1}K_2^{\frac{p}{2}}+c_2(p)K_2\right]\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du.$$

Proof of Theorem 5

Similarly, we obtain

$$
\begin{aligned}
&\mathbf{E}\{\sup_{s\le t\le s+\delta}[\int_s^t\int_{|y|\ge c}(|x(u-)+I(x(u-),y)|^p-|x(u-)|^p)\tilde{N}(du,dy)\\
&\quad+\int_s^t\int_{|y|\ge c}(|x(u-)+I(x(u-),y)|^p-|x(u-)|^p\\
&\quad-p|x(u-)|^{p-2}x^T(u-)H(x(u-),y))\nu(dy)du]\}\\
&\le c_1(p)\mathbf{E}\left[\left(\int_s^{s+\delta}\int_{|y|<c}|H(x(u-),y)|^2\nu(dy)du\right)^{\frac{p}{2}}\right]\\
&\quad+c_2(p)\mathbf{E}\left[\left(\int_s^{s+\delta}\int_{|y|<c}|H(x(u-),y)|^p\nu(dy)du\right)\right]
\end{aligned}
$$

Proof of Theorem 5

$$\leq c_1(p)\delta^{\frac{p}{2}-1}\mathbf{E}\left[\int_s^{s+\delta}\left(\int_{|y|<c}|H(x(u-),y)|^2\nu(dy)\right)^{\frac{p}{2}}du\right]$$

$$+c_2(p)\mathbf{E}\left[\left(\int_s^{s+\delta}\int_{|y|<c}|H(x(u-),y)|^p\nu(dy)du\right)\right]$$

$$\leq\left[c_1(p)\delta^{\frac{p}{2}-1}K_2^{\frac{p}{2}}+c_2(p)K_2\right]\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du. \tag{14}$$

Proof of Theorem 5

Thus,

$$\mathbf{E}\left(\sup_{s\le v\le s+\delta}|x(v)|^p\right)$$

$$\le \mathbf{E}|x(s)|^p + [K_1 p + \frac{1}{2}p(p-2)K_1 + \frac{p}{2}K_1^2]\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du$$

$$+\frac{1}{2}\mathbf{E}\left[\sup_{s\le v\le s+\delta}|x(v-)|^p\right] + 16p^2K_1\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du$$

$$+\left[c_1(p)\delta^{\frac{p}{2}-1}K_2^{\frac{p}{2}} + c_2(p)K_2\right]\int_s^{s+\delta}\mathbf{E}|x(u-)|^p du,$$

Proof of Theorem 5

$$\mathbf{E}\left(\sup_{s\le v\le s+\delta}|x(v)|^p\right)$$
$$\le 2\mathbf{E}|x(t)|^p+[2K_1p+p(p-2)K_1+pK_1^2+32p^2K_1$$
$$+2c_1(p)\delta^{\frac{p}{2}-1}K_2^{\frac{p}{2}}+2c_2(p)K_2]\delta\mathbf{E}\left(\sup_{t\le v\le t+\delta}|x(v)|^p\right). \quad (15)$$

Now choose the constant δ enough small such that

$$[2K_1p+p(p-2)K_1+pK_1^2+32p^2K_1+2c_1(p)\delta^{\frac{p}{2}-1}K_2^{\frac{p}{2}}+2c_2(p)K_2]\delta<1.$$

Proof of Theorem 5

Then, it follows from (15) that

$$\mathbf{E}\left(\sup_{t\le v\le t+\delta}|x(v)|^p\right)\le\frac{2}{1-C(p,\delta,K_1,K_2)}\mathbf{E}|x(t)|^p,\qquad(16)$$

where $C(p,\delta,K_1,K_2)=$
$[2K_1p+p(p-2)K_1+pK_1^2+32p^2K_1+2c_1(p)\delta^{\frac{p}{2}-1}K_2^p+2c_2(p)K_2^p]\delta$.
Hence,

$$\begin{aligned}&\sum_{m=1}^{\infty}\int_{\frac{(m-1)\delta}{2}}^{\frac{m\delta}{2}}\mathbf{E}\left(\sup_{t\le v\le t+\delta}|x(v)|^p\right)dt\\ &\le\frac{2}{1-C(p,\delta,K_1,K_2)}\sum_{m=1}^{\infty}\int_{\frac{(m-1)\delta}{2}}^{\frac{m\delta}{2}}\mathbf{E}|x(t)|^p dt\\ &=\frac{2}{1-C(p,\delta,K_1,K_2)}\int_0^{\infty}\mathbf{E}|x(t)|^p dt<\infty,\end{aligned}\qquad(17)$$

Proof of Theorem 5

which yields

$$\lim_{m\to\infty}\int_{\frac{(m-1)\delta}{2}}^{\frac{m\delta}{2}} \mathbf{E}\left(\sup_{t\le v\le t+\delta}|x(v)|^p\right)dt = 0. \tag{18}$$

Observe that

$$\mathbf{E}\left(\sup_{\frac{m\delta}{2}\le v\le\frac{(m+1)\delta}{2}}|x(v)|^p\right) \le \mathbf{E}\left(\sup_{t\le v\le t+\delta}|x(v)|^p\right)\ \forall t\in\left[\frac{(m-1)\delta}{2},\frac{m\delta}{2}\right].$$

Proof of Theorem 5

Then, by (18) we get

$$
\begin{aligned}
0 &\le \frac{\delta}{2} \lim_{m\to\infty} \mathbf{E}\left(\sup_{\frac{m\delta}{2}\le v\le \frac{(m+1)\delta}{2}} |x(v)|^p \right) \\
&= \lim_{m\to\infty} \int_{\frac{(m-1)\delta}{2}}^{\frac{m\delta}{2}} \mathbf{E}\left(\sup_{\frac{m\delta}{2}\le v\le \frac{(m+1)\delta}{2}} |x(v)|^p \right) dt \\
&\le \lim_{m\to\infty} \int_{\frac{(m-1)\delta}{2}}^{\frac{m\delta}{2}} \mathbf{E}\left(\sup_{t\le v\le t+\delta} |x(v)|^p \right) dt = 0,
\end{aligned}
$$

Proof of Theorem 5

which implies

$$\lim_{m\to\infty} \mathbf{E}\left(\sup_{\frac{m\delta}{2} \le v \le \frac{(m+1)\delta}{2}} |x(v)|^p \right) = 0. \tag{19}$$

Obviously, from (19) it follows that

$$\lim_{v\to\infty} \mathbf{E}|x(v)|^p = 0.$$

Therefore, the trivial solution of (2) is asymptotically stable in the pth moment.

Main results

Theorem 6 Suppose that the conditions of Theorem 5 hold. Then the trivial solution of (2) is almost surely asymptotically stable.

Main results

We now investigate the continuity of the trivial solution for system (2).

Theorem 7 Under Assumptions 1, 2 and 5, the trivial solution of (2) is continuous in the pth moment if $\mathbf{E}|x(t)|^p$ is bounded for any $t > 0$, i.e. there exists a positive constant K such that $\mathbf{E}|x(t)|^p \leq K < \infty$.

Proof.

$$\begin{aligned}\mathbf{E}|x(t+\delta)-x(t)|^p = \mathbf{E}|&\int_t^{t+\delta} f(x(u))du + \int_t^{t+\delta} g(x(u))dB(u)\\ &+ \int_t^{t+\delta}\int_{|y|<c} h_1(x(u-),y)\tilde{N}(du,dy)\\ &+ \int_{|y|\geq c} h_2(x(t-),y)N(dt,dy)|^p\end{aligned}$$

Proof of Theorem 7

$$
\begin{aligned}
&\leq 4^{p-1}\mathbf{E}\left|\int_t^{t+\delta} f(x(u-))du\right|^p + 4^{p-1}\mathbf{E}\left|\int_t^{t+\delta} g(x(u-))dB(u)\right|^p \\
&\quad + 4^{p-1}\mathbf{E}\left|\int_t^{t+\delta}\int_{|y|<c} h_1(x(u-),y)\tilde{N}(du,dy)\right|^p \\
&\quad + 4^{p-1}\mathbf{E}\left|\int_t^{t+\delta}\int_{|y|\geq c} h_2(x(u-),y)N(du,dy)\right|^p
\end{aligned}
$$

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